An Introduction to Map Theory

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Overview

- Why another foundational logic?
- Properties of map theory
- Equality in map theory
- Classical maps
- Modeling set theory

Problems with Existing Logics

Set theory is completely declarative and, in general, makes a mess of functions.

Higher order logic is classical and handles (total) functions easily. But it is much weaker than set theory and therefore some classical constructs can only with difficulty be formalized in higher order logic, if at all.

Constructive type theory is, well, constructive and hence not classical.

All of them have problems with general recursive functions.

Solution: Map Theory

The pros and cons of map theory are that

+ it is as strong as set theory.
+ it has the untyped lambda calculus, and thus general recursive functions, built in.
+ it is possible to give a quite natural representation of (ZFC) sets in map theory.
– it is very recent; not much practical work has been reported using it.
+/– it is untyped.

Basic Syntax

The terms (maps) Λ of map theory include

Λ	$\ni$	x	variable
	$\mid$	$\lambda x.A$	λ-abstraction
	$\mid$	$A`B$	application
	$\mid$	N	nil
	$\mid$	$\textbf{ifnil}\ A\ \textbf{then}\ B\ \textbf{else}\ C$	conditional
	$\mid$	$\varepsilon x.A$	choice

The judgments of map theory are equalities between terms, eg.,

$$\textbf{ifnil}\ \mathsf{N}\ \textbf{then}\ B\ \textbf{else}\ C = B$$

Some Prominent Maps

$$\begin{array}{lcll} \mathsf{I} & \stackrel{\text{def}}{=} & \lambda x.x & \text{Identity} \\ \mathsf{K} & \stackrel{\text{def}}{=} & \lambda x.\lambda y.x & \text{K-combinator} \\ \mathsf{Y} & \stackrel{\text{def}}{=} & \lambda f.(\lambda x.f`(x`x))`(\lambda x.f`(x`x)) & \text{Fixed Point Operator} \\ \perp & \stackrel{\text{def}}{=} & (\lambda x.x`x)`(\lambda x.x`x) & \text{Bottom} \end{array}$$

Depending on the context, $\perp$ will be thought of as either "undefined" or "divergent".

Some Properties of Map Theory

Laziness: The maps $\lambda x.\bot$ and $\bot$ are distinguishable.

Quartum non datur: Any map is either nil, equal to a lambda abstraction, or undefined; There is no fourth possibility.

Extensionality: Equality in map theory is *not* the same as β-convertibility.

Monotonicity: All maps are monotone wrt. the ordering given by

- $\bot \preceq \mathsf{N}$
- $\bot \preceq \lambda x.A$
- $\lambda x.A \preceq \lambda x.B$ if and only if, for all C, $A`C \preceq B`C$

Equality

We define $\cdot_R$ by

$$\begin{array}{rcl} \mathsf{N}_R & = & \mathsf{N} \\ (\lambda x.A)_R & = & \lambda \\ \bot_R & = & \bot \end{array}$$

Now, two maps A and B are equal in map theory if

$$(A`C_1`\cdots`C_n)_R = (B`C_1`\cdots`C_n)_R$$

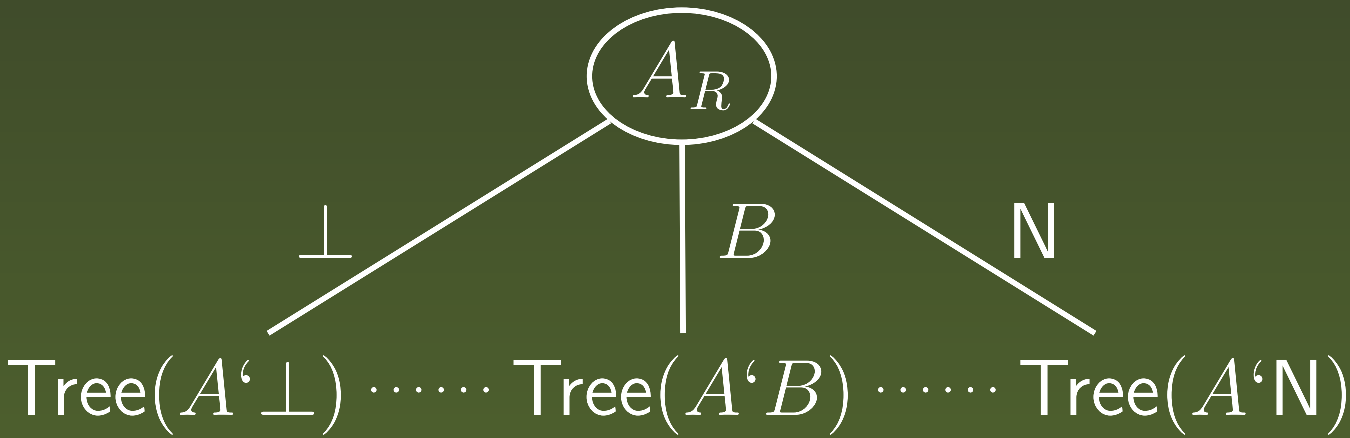
for all n and $C_1, \ldots, C_n$.

Map Trees

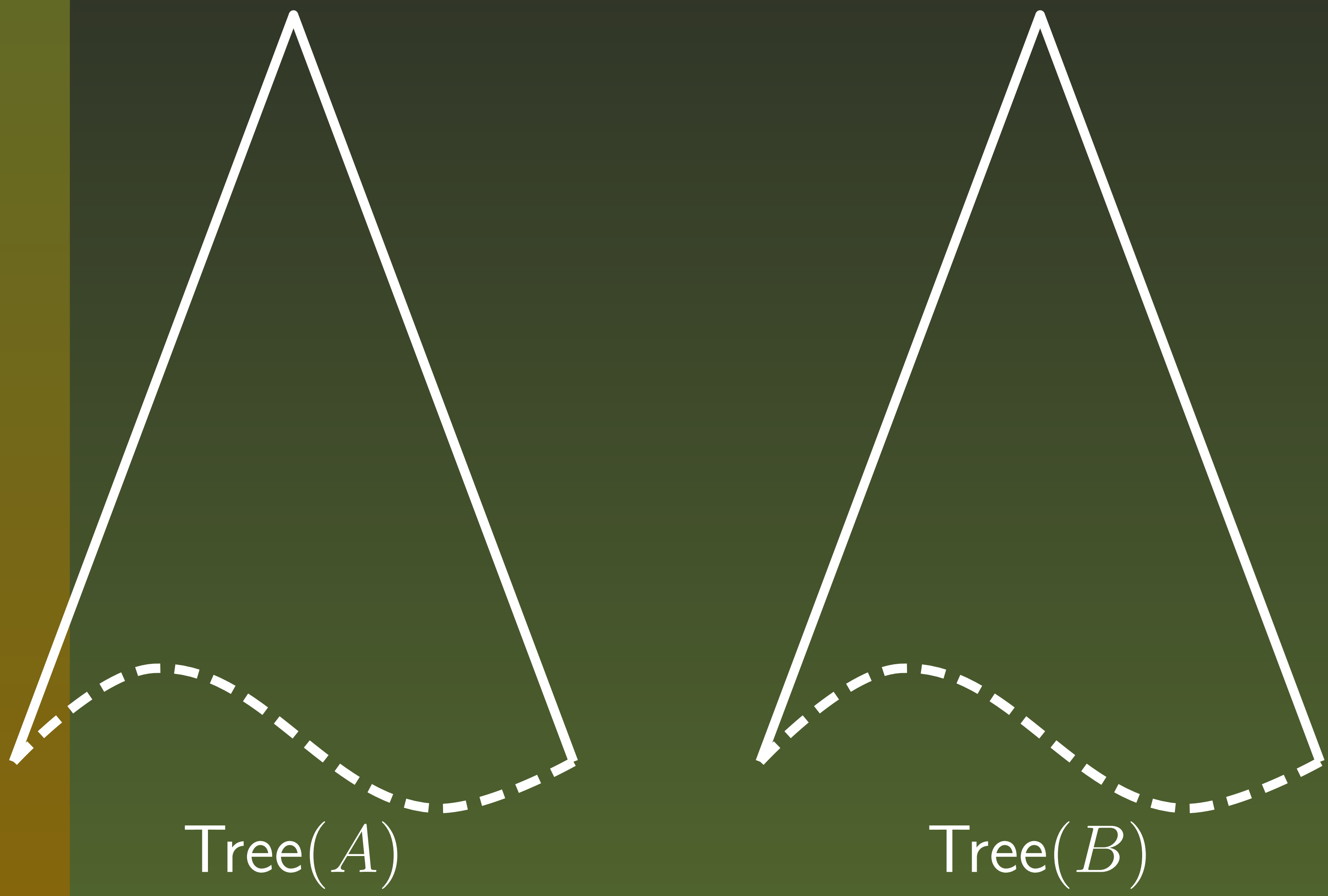
The map tree $\mathsf{Tree}(A)$ for the map A is constructed by:

- The root is A_R.
- For each map B, there is a branch, labeled B, from the root to the map tree of $A`B$.

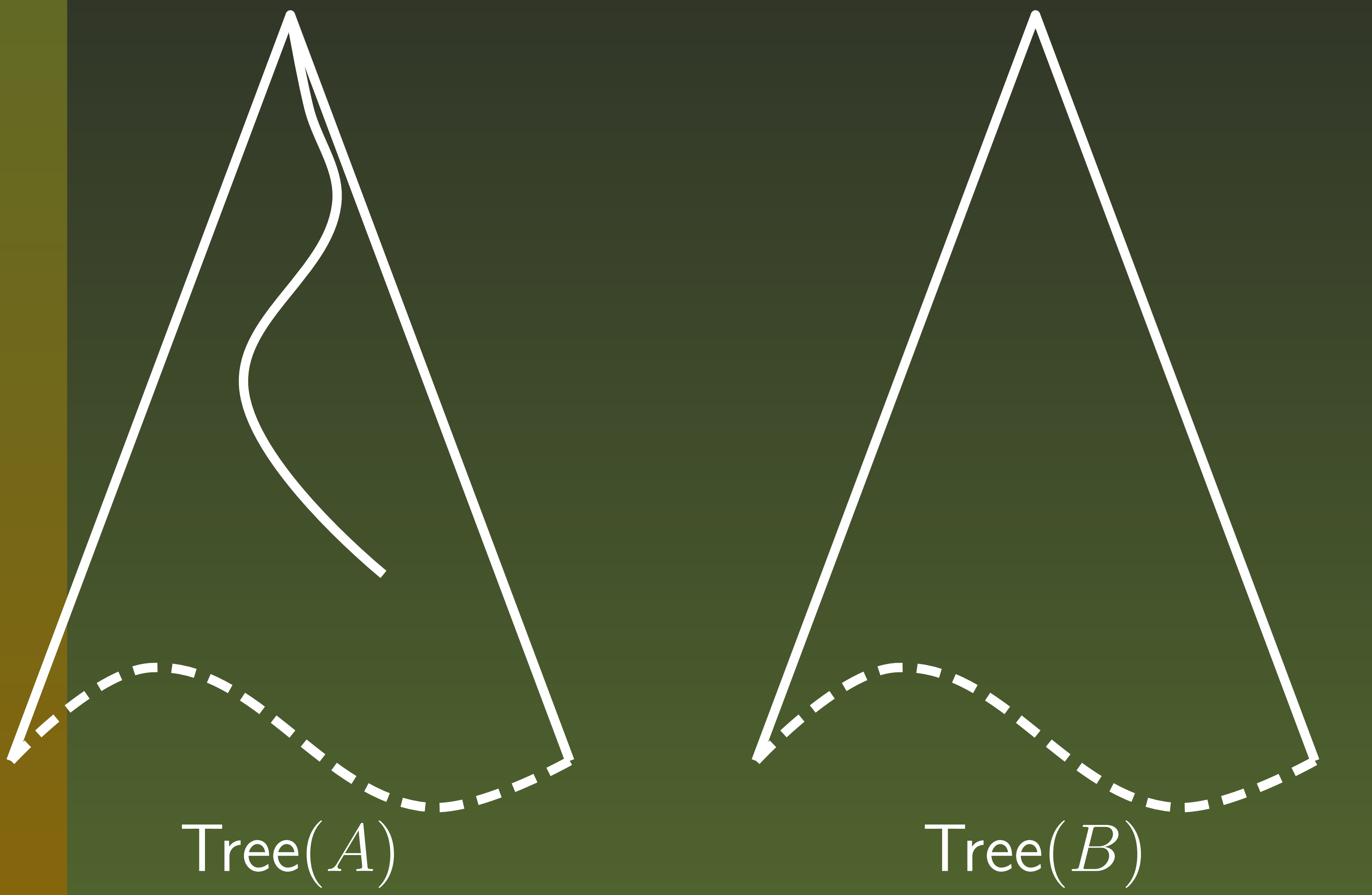
Graphically:



Intuition: Equality



Intuition: Equality



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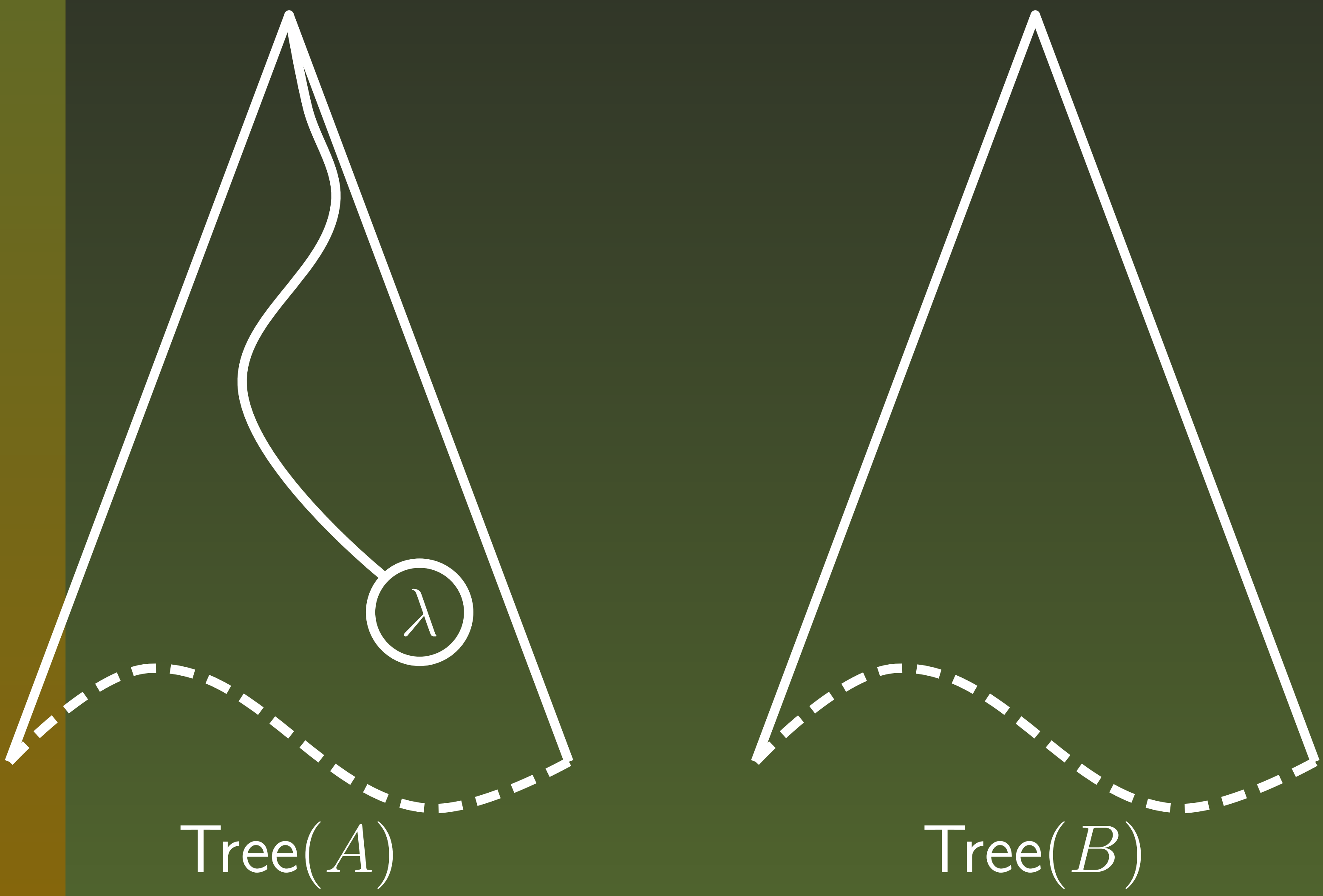
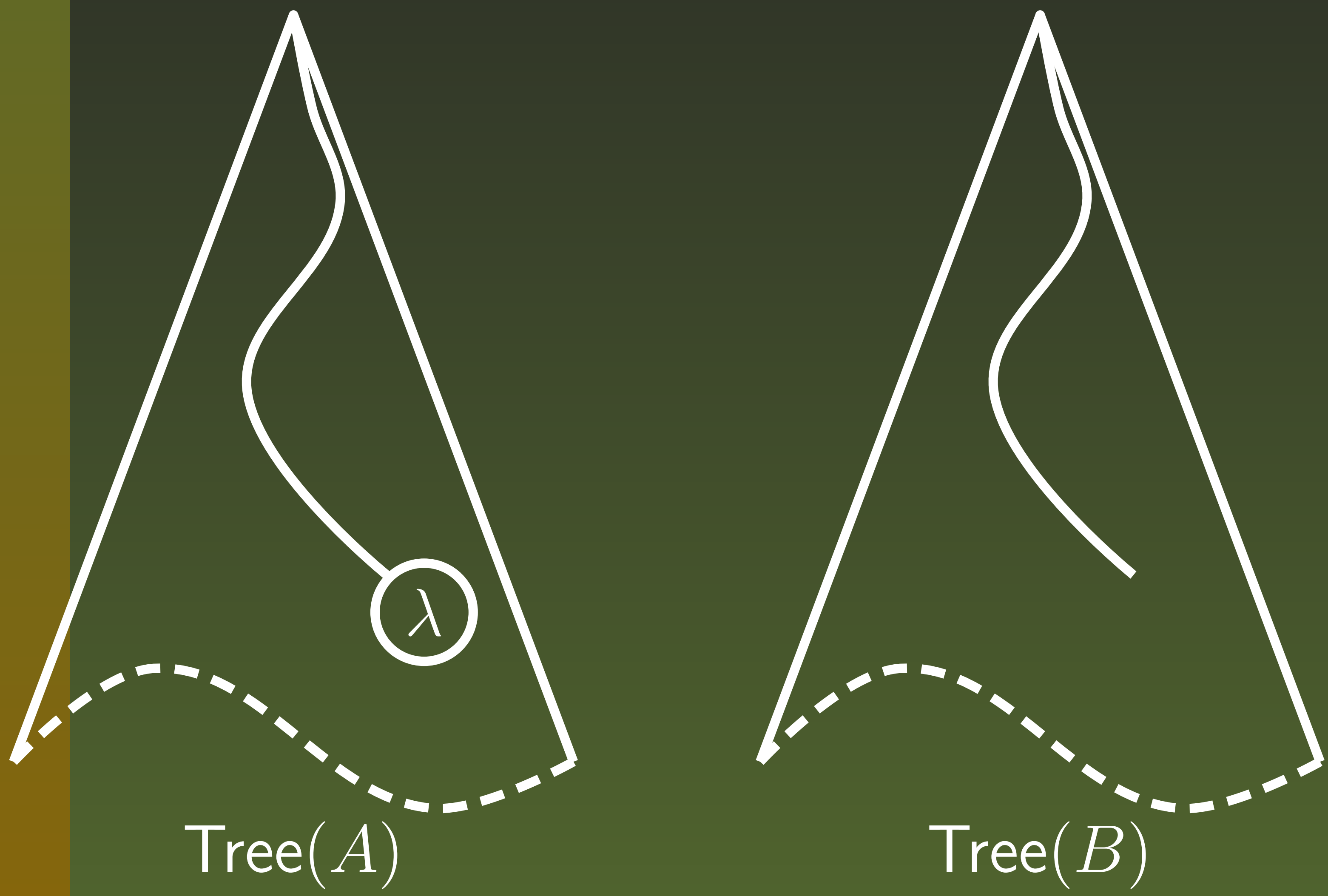
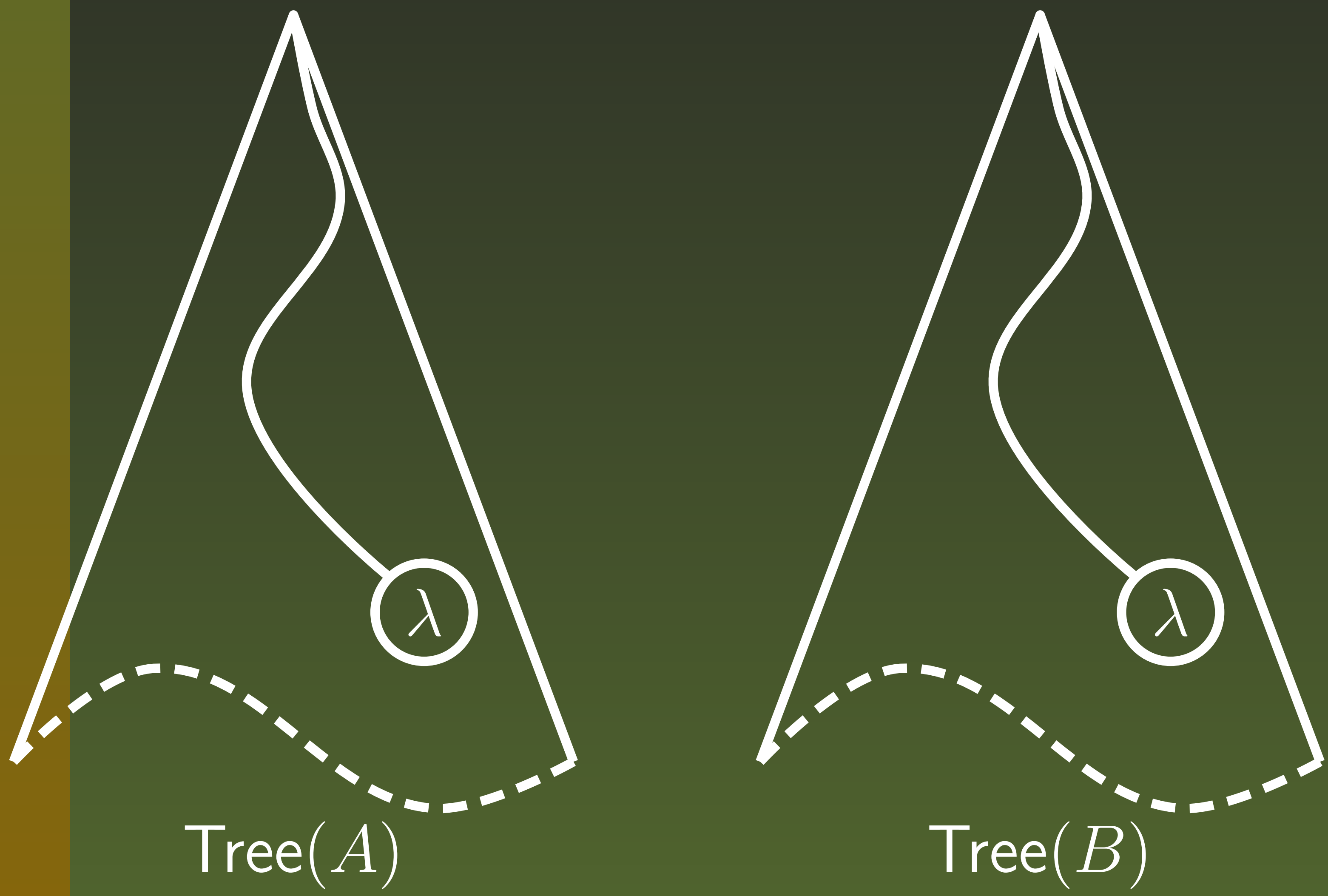


Intuition: Equality



Intuition: Equality



Extensionality

The axiom of Extensionality states that if

$$(A`X)_R = (B`X)_R$$

for all X and there exists a map C such that

$$\begin{aligned} A`X`Y &= A`(C`X`Y) \\ B`X`Y &= B`(C`X`Y) \end{aligned}$$

for all X and Y then

$$A`X = B`X$$

for all X.

Equality is not definable as a term

Assume

$$\mathsf{eq}\ A\ B = \mathsf{True} \quad \Longleftrightarrow \quad A = B$$

then

$$\mathsf{True} = \mathsf{eq}\ \bot\ \bot \preceq \mathsf{eq}\ A\ B$$

for all A and B.
Since True is maximal, this implies

$$\mathsf{eq}\ A\ B = \mathsf{True}$$

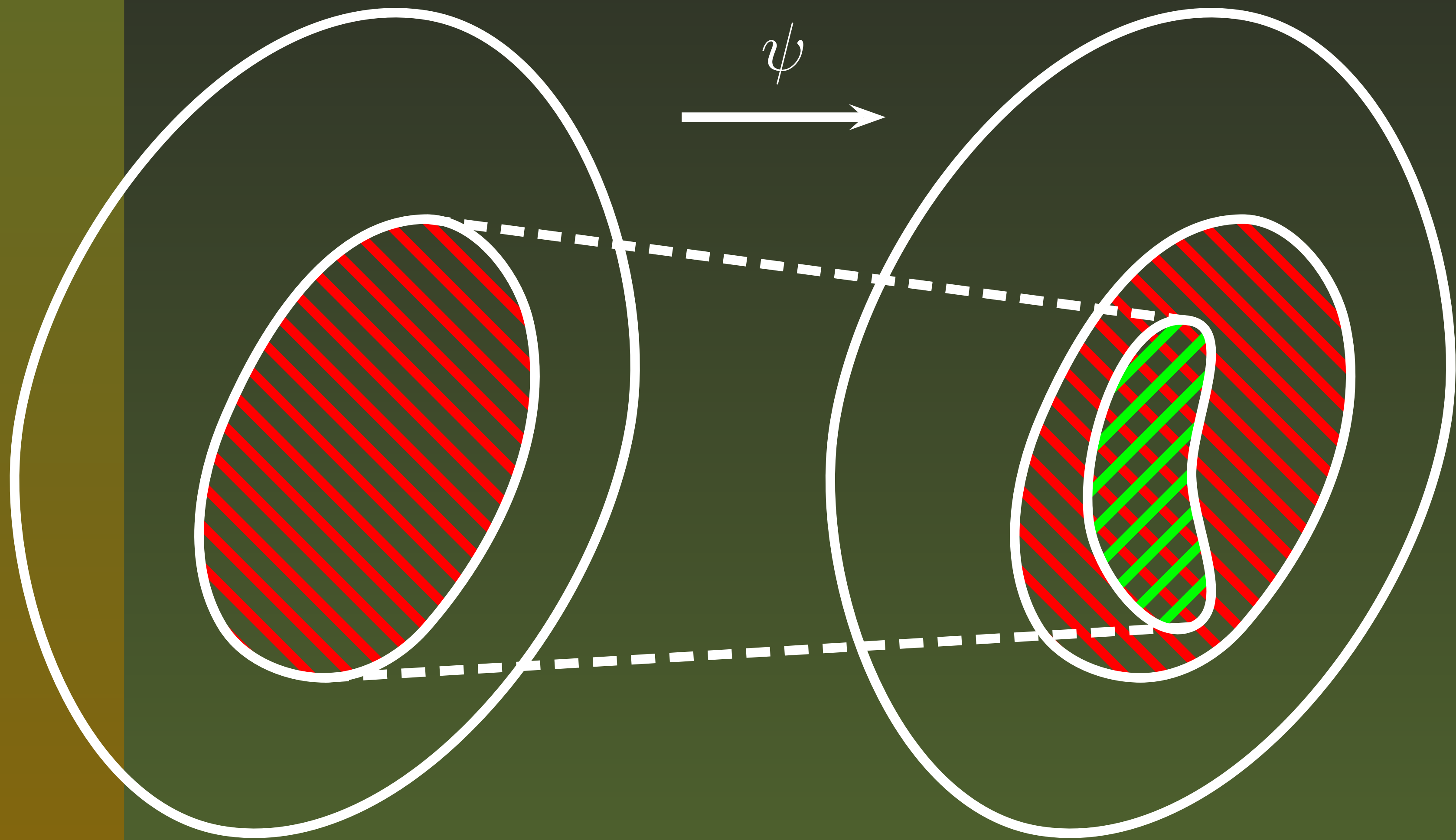
for all A and B.

Classical Maps

The classical maps of map theory correspond to the well-founded sets of set theory, and will also be used to model sets in map theory.
The class $\mathfrak{C}$ of classical maps obeys:

- Nil is classical.
- It is closed under application.
- Repeated application on classical maps will eventually return nil.
- The image of $\mathfrak{C}$ under a classical map is “small enough” to be considered a set.
- It is rich enough to model the sets of ZFC set theory.

Intuition: Classical Map



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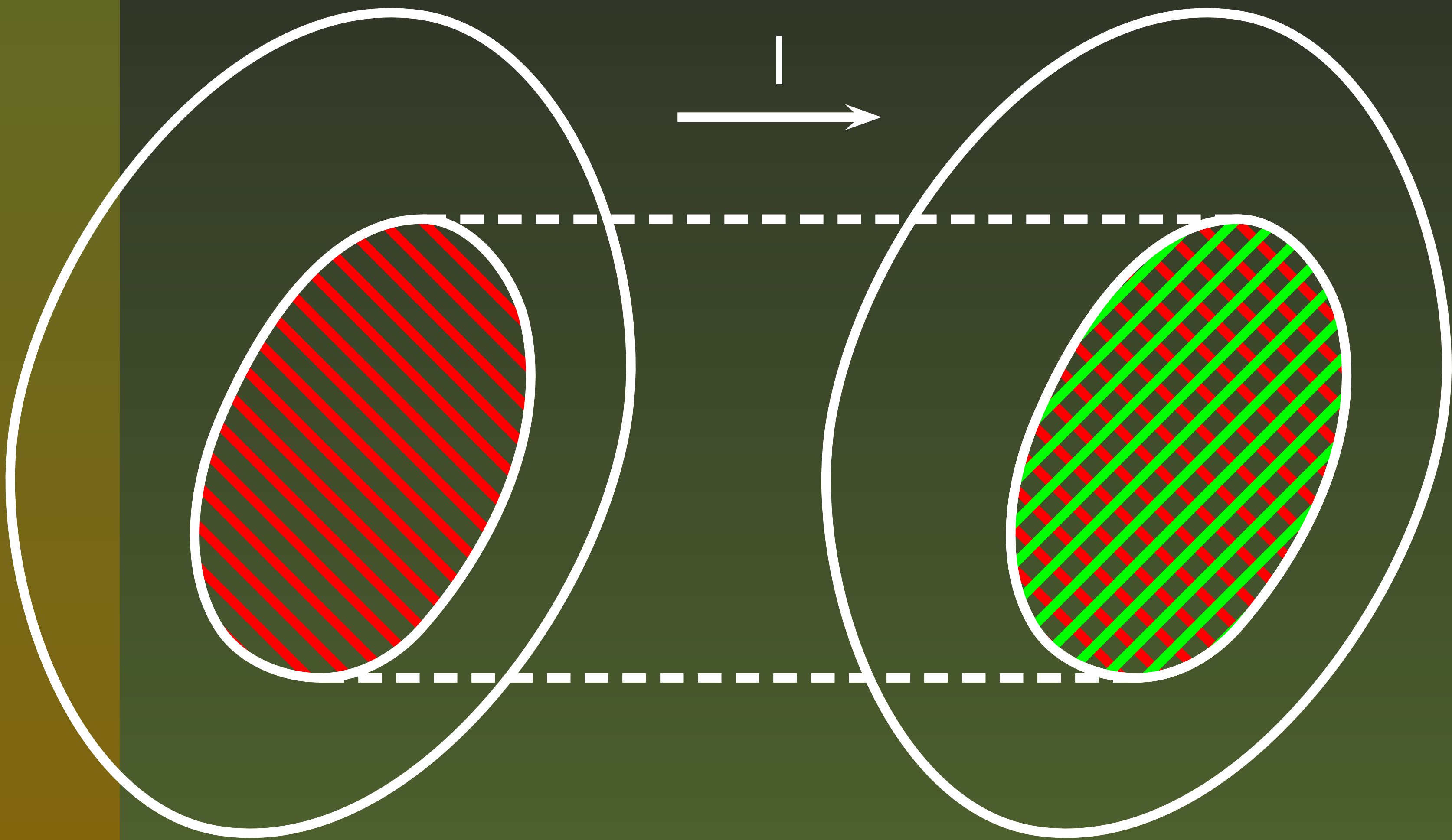
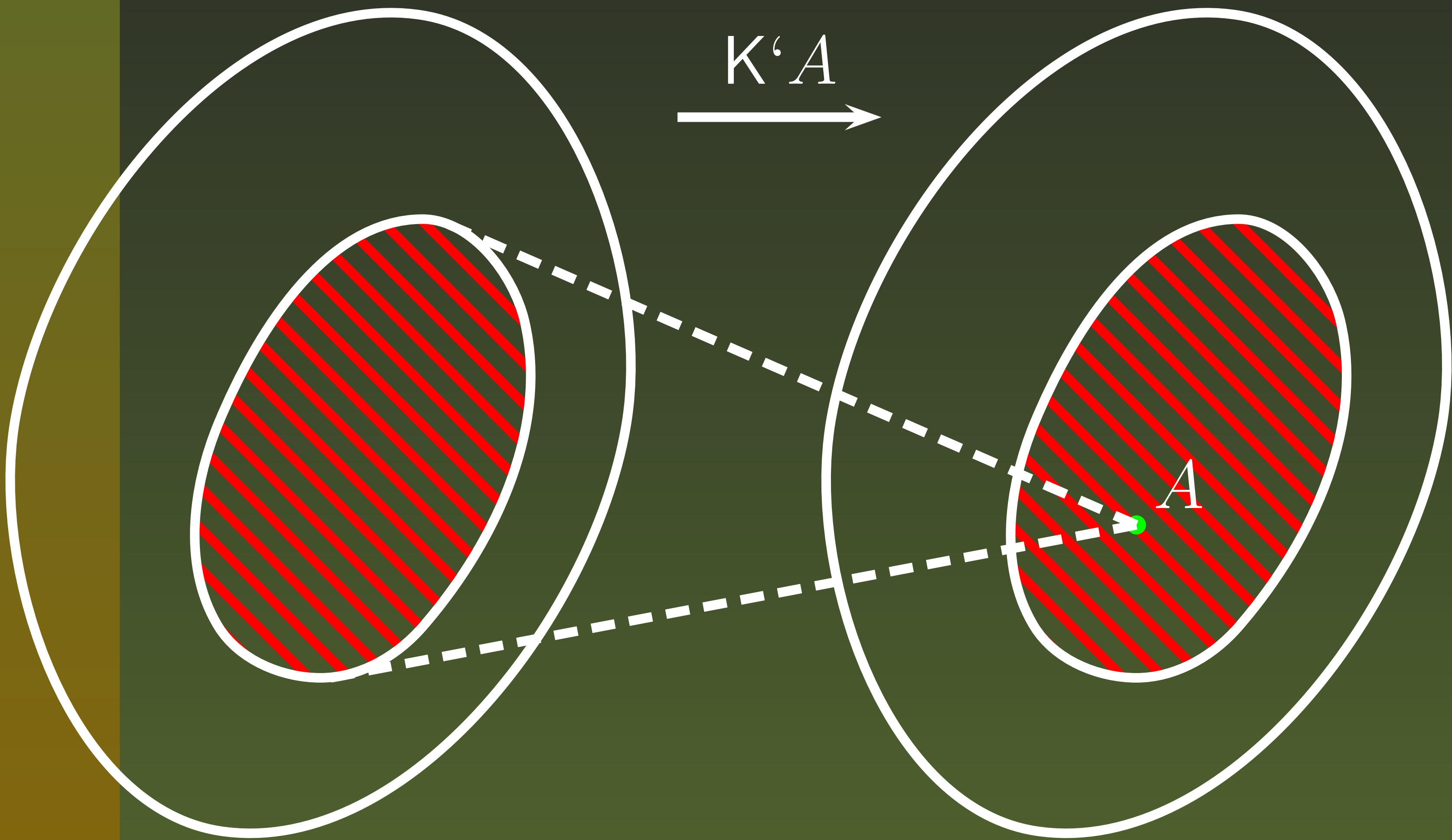


Intuition: Classical Map



Quantifiers in Map Theory

Unlike the universal quantifier of higher order logic, which is truly universal (for each type), the universal quantifier of map theory only ranges over the classical maps. It is defined using the choice-operator, which obeys:

- If $(\lambda x.A)`B$ is undefined for *some* classical B, then $\varepsilon x.A$ is undefined.
- If $(\lambda x.A)`B$ is defined for *all* classical B, then $\varepsilon x.A$ is classical.
- If $(\lambda x.A)`B$ is defined for *all* classical $B$ and there exists a classical $B$ such that $(\lambda x.A)`B$ is true, then $\varepsilon x.A$ is such a B.

Sets in Map Theory

We model the empty set $\emptyset$ by nil. Further, any other classical map A represents the set

$$\{A`B \mid B \in \mathfrak{C}\}$$

The equality $A =_{\mathsf{set}} B$ on sets is defined recursively as:

$$\textbf{ifnil } A \begin{cases} \textbf{ifnil } B \begin{cases} \mathsf{True} \\ \mathsf{False} \end{cases} \\ \textbf{ifnil } B \begin{cases} \mathsf{False} \\ (\forall x.\exists y.A`x =_{\mathsf{set}} B`y) \land \\ (\forall x.\exists y.A`y =_{\mathsf{set}} B`x) \end{cases} \end{cases}$$

Some Operations on Sets

$$A \in B \stackrel{\text{def}}{=} \textbf{ifnil } B \textbf{ then } \mathsf{False} \textbf{ else } \exists x. A =_{\mathsf{set}} B`x$$

$$\{A, B\} \stackrel{\text{def}}{=} \lambda x. \textbf{ifnil } x \textbf{ then } A \textbf{ else } B$$

$$\iota A \stackrel{\text{def}}{=} \varepsilon x. x \in A$$

Concluding Remarks

- Map theory shows promise as a foundational logic for both mathematics and computer science. It has general recursion built in and a quite natural embedding of set theory.
- The original version of map theory is on firm theoretical ground, while the new version of map theory has yet to be given a set theoretic model.
- A prototype proof system, Isabelle/MT, has been developed, and the model of ZFC has been verified within it.

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